

# Negative truncation of $L(\frac{1}{2}, \chi_d)$

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## Abstract

We prove that there exist large fundamental discriminants  $d$ , and partial sums of the Dirichlet series for  $L(\frac{1}{2}, \chi_d)$  having length a power of  $d$ , for which the partial sum is large and negative.

## 1 Introduction

As is well known, the central value  $L(\frac{1}{2}, \chi_d)$  of the  $L$ -function associated to fundamental discriminant  $d$  may be evaluated almost exactly by a smoothed sum of a little more than  $\sqrt{|d|}$  terms,

$$L\left(\frac{1}{2}, \chi_d\right) \doteq 2 \sum_{n \leq |d|^{\frac{1}{2} \log |d|}} \frac{\chi_d(n)}{\sqrt{n}} V\left(\sqrt{\frac{\pi}{|d|}} n\right).$$

Here we may choose the test function  $V$  on  $\mathbb{R}^+$  to satisfy  $V(0) = 1$ ,  $V \geq 0$  and, for all  $A > 0$ ,  $V(x) \ll_A (1+x)^{-A}$ . Since the Generalized Riemann Hypothesis implies that the central value is non-negative, this is some indication of a bias among small  $n$  toward  $\chi_d(n) = 1$ . Further evidence, again on GRH, is that for fixed  $\sigma > \frac{1}{2}$  and  $0 < a < 1/2$  we have positivity of the much shorter smoothed sums

$$\sum_n \frac{\chi_d(n)}{n^\sigma} \phi\left(\frac{n}{|d|^a}\right) \geq 0, \quad \phi(0) = 1, \quad \phi \in C_0^\infty(\mathbb{R})$$

for all sufficiently large  $|d|$ . Again, this is because on GRH the partial sum is a strong approximation to the  $L$ -value  $L(\sigma, \chi_d)$ .

At the central point one should not hope to approximate  $L(\frac{1}{2}, \chi_d)$  with a partial sum of its Dirichlet series of length shorter than  $\sqrt{d}$ . Nonetheless, we might wonder whether positivity holds for sums of length a power of the conductor,

$$0 < a < \frac{1}{2} : \quad \sum_{n \leq |d|^a} \frac{\chi_d(n)}{\sqrt{n}} \geq -o(1)?$$

We show that this is not necessarily the case, and that there exist smoothed truncations that take large negative values of magnitude nearly as large as the largest positive extreme values known for  $L(\frac{1}{2}, \chi_d)$  [3].

**Theorem 1.1.** *Let  $D$  be large. Let  $\phi$  be a smooth test function on  $\mathbb{R}^+$  satisfying  $\phi \equiv 1$  on a neighborhood of 0, and  $\phi \equiv 0$  for  $x > 2$ . Let  $0 < a < \frac{2}{9}$ . There exists  $c = c(a) > 0$  and fundamental discriminant  $d$ ,  $4D \leq |d| \leq 8D$  such that*

$$\sum_n \frac{\chi_d(n)}{\sqrt{n}} \phi\left(\frac{n}{|D|^a}\right) \leq -\exp\left((c(a) + o(1))\sqrt{\frac{\log D}{\log \log D}}\right).$$

As in [3] we translate the problem of producing an omega result into estimation of a ratio of forms in the coefficients of an auxilliary multiplicative function. But whereas in [3] a large ratio results from sums taken over smooth numbers composed of only small prime factors, here the sum over smooth numbers is quite small, and instead we use large primes to amplify fluctuations in partial sums of the smaller coefficients. The fact that the large primes must make a non-trivial contribution is what prevents the argument from producing a similar result at a fixed point to the right of the half-line.

## 2 Initial steps

For odd squarefree  $d > 0$  introduce

$$R(d) = \sum_{n \leq Z} r(n) \chi_{8d}(n)$$

with  $r$  a real-valued multiplicative function supported on odd squarefree integers and  $Z$  a parameter. Evidently there exists fundamental discriminant  $8d^*$  with

$$\begin{aligned} & \sum_n \frac{\chi_{8d^*}(n)}{\sqrt{n}} \phi\left(\frac{n}{x}\right) \\ & \leq \sum_{\frac{D}{2} < d \leq D} \mu(2d)^2 R(d)^2 \sum_n \frac{\chi_{8d}(n)}{\sqrt{n}} \phi\left(\frac{n}{x}\right) \Bigg/ \sum_{\frac{D}{2} < d \leq D} \mu(2d)^2 R(d)^2 \\ & = \mathcal{N}/\mathcal{D} \end{aligned}$$

so it will suffice to produce a Dirichlet polynomial  $R$  for which

$$\mathcal{N}/\mathcal{D} \leq -\exp\left((c(a) + o(1))\sqrt{\frac{\log D}{\log \log D}}\right).$$

### Parameters

Set  $\delta = \frac{2}{9} - a$ ,  $x = D^a$ ,  $Z = \min(xD^\delta, x^{3/2})$  and also

$$Y = \sqrt{\frac{Z}{x}} = \min(D^{\delta/2}, x^{1/4}), \quad L = \sqrt{\log Y \log \log Y}.$$

We define  $r$  separately at small and large primes. Thus let  $\mathcal{P}^-$  be the primes in the interval  $L^{5\pi/3} \leq p < L^{7\pi/3}$  and set

$$r_-(p) = \begin{cases} \cos\left(\frac{\log p}{\log(L^2)}\right) \frac{L}{\sqrt{p} \log p} & p \in \mathcal{P}^- \\ 0 & \text{otherwise} \end{cases}.$$

The definition of  $r_+$  is less explicit since it will depend upon fluctuations in partial sums of  $r_-$ . We let  $\mathcal{P}^+$  be the set of primes in an interval  $[\frac{B}{4}, B]$ , requiring for now only that  $x^{1-\epsilon} < B \leq x$ . We let

$$r_+(p) = \frac{\epsilon_p}{\sqrt{p}(\log x)^2}$$

for  $p \in \mathcal{P}^+$  where the sign  $\epsilon_p = \pm 1$  will be set later. The function  $r$  is now defined by  $r(n) = r_- * r_+(n)$ , so that

$$r(p) = \begin{cases} \cos\left(\frac{\log p}{\log(L^2)}\right) \frac{L}{\sqrt{p} \log p} & p \in \mathcal{P}^- \\ \frac{\epsilon_p}{\sqrt{p}(\log x)^2} & p \in \mathcal{P}^+ \\ 0 & p \notin \mathcal{P}^- \cup \mathcal{P}^+. \end{cases}.$$

We record some simple properties of  $r$  in the following lemma.

**Lemma 2.1.** *We have the bound*

$$\sum_n \frac{|r(n)|d(n)}{\sqrt{n}} = \prod_p \left(1 + \frac{2|r(p)|}{\sqrt{p}}\right) \leq \exp\left(O\left(\sqrt{\frac{\log D}{\log \log D}}\right)\right). \quad (1)$$

*We also have the following truncation properties.*

*Let  $f(n)$  be any multiplicative function satisfying the bound at primes  $|f(p)| \leq 2$ . Let  $M_1 \geq \sqrt{Y}$ . There exists  $c_1 > 0$  such that*

$$\left| \sum_{\substack{n > M_1 \\ (n, \mathcal{P}^+) = 1}} \frac{r(n)f(n)}{\sqrt{n}} \right| \leq \exp\left(-c_1 \frac{\log D}{(\log \log D)^2}\right). \quad (2)$$

*Let  $M_2 \geq Y$ , let  $\ell \geq 1$  be an integer and let  $g(n)$  be multiplicative, satisfying  $0 \leq g(n) \leq 1$ . There exists  $c_2 > 0$  such that*

$$\sum_{\substack{n \leq M_2 \\ (\ell, n) = 1 \\ (n, \mathcal{P}^+) = 1}} r(n)^2 g(n) = \left(1 + O\left(\exp\left(-c_2 \frac{\log D}{(\log \log D)^2}\right)\right)\right) \prod_{\substack{p \in \mathcal{P}^- \\ p \nmid \ell}} (1 + r(p)^2 g(p)). \quad (3)$$

*Proof.* The bound (1) is immediate from the prime number theorem. The two truncation bounds use ‘Rankin’s trick’. Set  $\alpha = \frac{1}{(\log \log D)^2}$ . Then

$$\begin{aligned} \left| \sum_{\substack{n > M_1 \\ (n, \mathcal{P}^+) = 1}} \frac{r(n)f(n)}{\sqrt{n}} \right| &\leq M_1^{-\alpha} \sum_{(n, \mathcal{P}^+) = 1} \frac{|r(n)|d(n)}{n^{\frac{1}{2}-\alpha}} = M_1^{-\alpha} \prod_{p \in \mathcal{P}^-} \left(1 + \frac{2|r(p)|}{p^{\frac{1}{2}-\alpha}}\right) \\ &\leq \exp\left(-\alpha \log M_1 + (2 + o(1)) \sum_{p \in \mathcal{P}^-} \frac{|r(p)|}{\sqrt{p}}\right) \end{aligned}$$

The bound now follows from  $\sum_{p \in \mathcal{P}^-} \frac{|r(p)|}{\sqrt{p}} = o(\sqrt{\log D})$  while  $\log M_1 \gg \log D$ .

For (3) write

$$\sum_{\substack{n \leq M_2 \\ (\ell, n)=1 \\ (n, \mathcal{P}^+)=1}} r(n)^2 g(n) = \prod_{\substack{p \in \mathcal{P}^- \\ p \nmid \ell}} (1 + r(p)^2 g(p)) + O \left( M_2^{-\alpha} \prod_{\substack{p \in \mathcal{P}^- \\ p \nmid \ell}} (1 + r(p)^2 g(p) p^\alpha) \right)$$

The relative error is

$$M_2^{-\alpha} \prod_{p \in \mathcal{P}^-} \frac{1 + r(p)^2 g(p) p^\alpha}{1 + r(p)^2 g(p)} \leq \exp \left( -\alpha \log M_2 + \sum_{p \in \mathcal{P}^-} r(p)^2 (p^\alpha - 1) \right)$$

and the bound now follows from

$$\sum_{p \in \mathcal{P}^-} r(p)^2 (p^\alpha - 1) \lesssim \alpha \sum_{p \in \mathcal{P}^-} r(p)^2 \log p \leq \alpha L^2 \sum_{L^{\frac{5\pi}{3}} \leq p \leq L^{\frac{7\pi}{3}}} \frac{1}{p \log p} \lesssim \frac{6}{5\pi} \alpha \log Y.$$

□

### Evaluation of forms

The following lemma describes the orthogonality of the characters  $\chi_{8d}$ .

**Lemma 2.2** (Orthogonality relation, [2] Lemma 3.1). *Let  $n$  be square. We have*

$$\sum_{D/2 < d \leq D} \mu^2(2d) \chi_{8d}(n) = \frac{3}{\pi^2} D \prod_{p|2n} \left( \frac{p}{p+1} \right) + O(D^{1/2+\epsilon} n^\epsilon).$$

If  $n$  is not square then

$$\sum_{d < D} \mu^2(2d) \chi_{8d}(n) = O(D^{1/2} n^{1/4} \log n).$$

Using this lemma, we now give asymptotic evaluations for  $\mathcal{D}$  and  $\mathcal{N}/\mathcal{D}$ .

**Lemma 2.3.** *We have the following asymptotic evaluation of  $\mathcal{D}$ .*

$$\mathcal{D} = \sum_{D/2 < d \leq D} \mu(2d)^2 R(8d)^2 \sim \frac{2}{\pi^2} D \sum_{n \leq R} r'(n)^2 \sim \frac{2}{\pi^2} D \prod_{p \in \mathcal{P}^-} (1 + r'(p)^2).$$

where  $r'(p) = r(p) \sqrt{\frac{p}{p+1}}$ .

*Proof.* Expanding the square,

$$\mathcal{D} = \sum_{n_1, n_2 \leq Z} r(n_1) r(n_2) \sum_{D/2 < d \leq D} \mu(2d)^2 \chi_{8d}(n_1 n_2).$$

Recall that  $r$  is supported on odd squarefrees. We obtain a diagonal term

$$\frac{2}{\pi^2} D \sum_{n \leq Z} r'(n)^2 = \frac{2}{\pi^2} D \sum_{n \leq Z, (n, \mathcal{P}^+)=1} r'(n)^2 + O \left( D \sum_{p \in \mathcal{P}^+} r'(p)^2 \sum_{n \leq Z/p} r'(n)^2 \right).$$

Since  $\sum_{p \in \mathcal{P}^+} r'(p)^2 = o(1)$  the error term is negligible compared to the main term. Since  $Z > Y$  the main term is  $\sim \frac{2}{\pi^2} D \prod_{p \in \mathcal{P}^-} (1 + r'(p)^2)$  by (3) of Lemma 2.1.

The off-diagonal terms are bounded by

$$\begin{aligned} &\ll D^{1/2+\epsilon} \sum_{n_1, n_2 < Z} |r(n_1)r(n_2)|(n_1 n_2)^{1/4} \\ &\ll D^{1/2+\epsilon} Z^{3/2} \sum_{n_1, n_2} \frac{|r(n_1)||r(n_2)|}{\sqrt{n_1 n_2}} \ll D^{1/2+\frac{3}{2}\delta+\epsilon} x^{3/2} \ll D^{5/6+\epsilon} \end{aligned}$$

by using  $x = D^{2/9-\delta}$ . □

The main evaluation of  $\mathcal{N}/\mathcal{D}$  is as follows.

**Proposition 2.4.** *We may express  $\mathcal{N}/\mathcal{D} \sim \Sigma_1 + O(\Sigma_2) + o(1)$ , where*

$$\begin{aligned} \Sigma_1 &= \frac{2}{(\log x)^2} \sum_{p \in \mathcal{P}^+} \frac{\epsilon_p}{p} \sum_{(\ell, \mathcal{P}^+)=1} \frac{\tilde{r}(\ell)d(\ell)}{\sqrt{\ell}} \sum_m \frac{b(m, \ell)}{m} \phi\left(\frac{\ell p m^2}{x}\right) \\ \Sigma_2 &= \sum_{(\ell, \mathcal{P}^+)=1} \frac{\tilde{r}(\ell)d(\ell)}{\sqrt{\ell}} \sum_m \frac{b(m, \ell)}{m} \phi\left(\frac{\ell m^2}{x}\right) \end{aligned}$$

Here  $b(m, \ell)$  and  $\tilde{r}(p)$  are given by

$$b(m, \ell) = \prod_{p|m, p \nmid 2\ell} \left( \frac{p}{p+1} \frac{1+r_-(p)^2}{1+r_-(p)^2 \frac{p}{p+1}} \right), \quad \tilde{r}(p) = \frac{r(p)}{(1+\frac{1}{p})(1+r'(p)^2)}.$$

*Proof.* Expand the square in  $\mathcal{N}$  and pass the sum over  $d$  inside to obtain

$$\mathcal{N} = \sum_{\ell_1, \ell_2 \leq Z} r(\ell_1)r(\ell_2) \sum_n \frac{\phi\left(\frac{n}{x}\right)}{\sqrt{n}} \sum_{D/2 < d \leq D} \mu^2(2d) \chi_{8d}(\ell_1 \ell_2 n)$$

Recall that  $r$  is supported on squarefrees. Pull out the gcd  $g = (\ell_1, \ell_2)$ , writing  $\ell'_1 = \frac{\ell_1}{g}$ ,  $\ell'_2 = \frac{\ell_2}{g}$ . We obtain a diagonal term coming from  $n$  of form  $\ell'_1 \ell'_2 m^2$ . This obtains

$$\frac{2D}{\pi^2} \sum_{\substack{\ell=\ell_1 \ell_2 \\ (\ell_1, \ell_2)=1}} \frac{r(\ell)}{\sqrt{\ell}} \sum_{\substack{g \leq Z/\max(\ell_1, \ell_2) \\ (g, \ell)=1}} r(g)^2 \sum_m \frac{\phi\left(\frac{\ell m^2}{x}\right)}{m} \prod_{\substack{p|\ell m g \\ \text{odd}}} \frac{p}{p+1}. \quad (4)$$

The off-diagonal terms are bounded by

$$\begin{aligned} &D^{1/2+\epsilon} \sum_{\ell_1, \ell_2 \leq Z} \sum_{n \leq x} \frac{|r(\ell_1)||r(\ell_2)|( \ell_1 \ell_2 )^{1/4}}{n^{1/4}} \\ &\ll D^{1/2+\epsilon} x^{3/4} Z^{3/2} \sum_{\ell_1, \ell_2} \frac{|r(\ell_1)r(\ell_2)|}{\sqrt{\ell_1 \ell_2}} \ll D^{\frac{1}{2}+\frac{3\delta}{2}+\epsilon} x^{\frac{9}{4}} = O(D^{1-\frac{3\delta}{4}+\epsilon}). \end{aligned}$$

Before proceeding further, we may comment that the diagonal sum (4) is bounded absolutely by (see (1))

$$\begin{aligned} &\ll D \log D \prod_{p \in \mathcal{P}^+ \cup \mathcal{P}^-} \left( \left( 1 + \frac{2|r(p)|}{\sqrt{p}} \right) (1 + |r'(p)|^2) \right) \\ &\leq \mathcal{D} \exp \left( O \left( \sqrt{\frac{\log D}{\log \log D}} \right) \right), \end{aligned}$$

so that, even though the sum contains terms of differing sign, we may make relative errors on the order of

$$(1 + O(\exp(-\sqrt{\log D})))$$

within individual terms without altering the final asymptotics.

Bearing this in mind, we split the diagonal term (4) into two sums  $\Sigma_1^0 + \Sigma_2^0$  according as  $\ell$  does or does not have a factor  $p \in \mathcal{P}^+$ . In the former case, we have

$$\Sigma_1^0 = \frac{2D}{\pi^2} \sum_{p \in \mathcal{P}^+} \frac{2r(p)}{\sqrt{p}} \sum_{\ell < 2x/p} \frac{r(\ell)}{\sqrt{\ell}} \sum_m \frac{\phi\left(\frac{\ell p m^2}{x}\right)}{m} \sum_{\ell_1 \ell_2 = \ell} \sum_{\substack{g \leq Z/p\ell_1 \\ (g, \ell)=1}} r(g)^2 \prod_{\substack{p' | \ell m g \\ \text{odd}}} \frac{p'}{p' + 1}$$

Notice that  $g < \frac{Z}{p\ell_1}$  implies  $g \leq x^{1/2+\epsilon}$  so that  $g$  has no factors from  $\mathcal{P}^+$ . Also

$$Z/p\ell_1 \geq Z/2x > Y,$$

so that (3) of Lemma 2.1 implies that to within admissible relative error,

$$\sum_{\substack{g \leq Z/p\ell_1 \\ (g, \ell)=1}} \left( r(g)^2 \prod_{\substack{p' | g \\ p' \nmid m}} \frac{p'}{p' + 1} \right) = \prod_{\substack{p \in \mathcal{P}^- \\ p \nmid m, p \nmid \ell}} (1 + r'(p)^2) \prod_{\substack{p \in \mathcal{P}^- \\ p | m, p \nmid \ell}} (1 + r(p)^2).$$

Substituting this evaluation into  $\Sigma_1^0$  and dividing by  $\mathcal{D}$  we obtain  $\Sigma_1$  from the Proposition.

It remains to treat the sum  $\Sigma_2^0$ , and this we do by splitting the sum further as  $\Sigma_2^0 = \Sigma_2^1 + \Sigma_2^2$ , depending on whether or not  $g$  has a factor from  $\mathcal{P}^+$ . We first handle the case that  $g$  does not contain such a factor, which we call  $\Sigma_2^1$ . We have

$$\Sigma_2^1 = \frac{2D}{\pi^2} \sum_{\substack{\ell < 2x \\ (\ell, \mathcal{P}^+)=1}} \frac{r(\ell)d(\ell)}{\sqrt{\ell}} \sum_{\substack{(g, \ell)=1 \\ (g, \mathcal{P}^+)=1}} r(g)^2 \sum_m \frac{\phi\left(\frac{\ell m^2}{x}\right)}{m} \prod_{\substack{p | \ell m g \\ \text{odd}}} \frac{p}{p + 1}.$$

Instead of the divisor function  $d(\ell)$  we should have included a sum over  $\ell_1 \ell_2 = \ell$ , with the restriction on  $g$  that  $g < Z/\max(\ell_1, \ell_2)$ , but this may be removed, again by Lemma 2.1. Writing the sum over  $g$  as a product and dividing by  $\mathcal{D}$  we arrive at  $\Sigma_2$ .

It remains to treat  $\Sigma_2^2$ . Here we have

$$\Sigma_2^2 = \frac{2D}{\pi^2} \sum_{p \in \mathcal{P}^+} r(p)^2 \sum_{\substack{\ell_1 \ell_2 = \ell < 2x \\ (\ell, \mathcal{P}^+)=1}} \frac{r(\ell)}{\sqrt{\ell}} \sum_m \frac{\phi\left(\frac{\ell m^2}{x}\right)}{m} \sum_{\substack{g \leq Z/p \max(\ell_1, \ell_2) \\ (g, \ell)=1}} r(g)^2 \prod_{\substack{p | \ell m g \\ \text{odd}}} \frac{p}{p + 1}.$$

We wish to again replace the sum over  $g$  with a product, but this is only valid if the sum is sufficiently long. Since  $\ell$  restricts the length of  $g$ , we first bound in absolute value the contribution of all terms with

$$\ell > \min(D^{\delta/4}, x^{1/8}) = \sqrt{Y}.$$

This is negligible:

$$\ll D(\log D)^{-2} \prod_p (1 + r(p)^2) \sum_{\substack{\ell > \sqrt{Y} \\ (\ell, \mathcal{P}^+) = 1}} \frac{|r(\ell)|d(\ell)}{\sqrt{\ell}} \ll \mathcal{D} \exp\left(-c \frac{\log D}{(\log \log D)^2}\right)$$

by applying (2) of Lemma 2.1.

Restricting to terms with  $\ell < \sqrt{Y}$ , we see that the condition  $g \leq \frac{Z}{p \max(\ell_1, \ell_2)}$  implies  $g \leq x^{1/2+\epsilon}$ , so  $g$  is free of prime factors from  $\mathcal{P}^+$ . Also,

$$\frac{Z}{p \max(\ell_1, \ell_2)} \geq \frac{Z}{p\sqrt{Y}} > Y^{3/2},$$

so that we may now replace the sum over  $g$  with a product by again applying (3) of Lemma 2.1.

Having done this, we may now reinsert those terms  $\sqrt{Y} < \ell < 2x$  that are composed solely of prime factors from  $\mathcal{P}^-$ , again with negligible error. With these adjustments, we now find that

$$\Sigma_2^2/\mathcal{D} = \Sigma_2 \times \sum_{p \in \mathcal{P}^+} r(p)^2,$$

and since  $\sum_{p \in \mathcal{P}^+} r(p)^2 = o(1)$ , the proof is complete.  $\square$

## Partial sums

The remainder of the proof of Theorem 1.1 is concerned with the partial sums

$$S(y) := \sum_{(\ell, \mathcal{P}^+) = 1} \frac{\tilde{r}(\ell)d(\ell)}{\sqrt{\ell}} \sum_m \frac{b(m, \ell)}{m} \phi\left(\frac{\ell m^2}{y}\right). \quad (5)$$

Let the associated Dirichlet series be

$$F(s) = \sum_{(\ell, \mathcal{P}^+) = 1} \sum_m \frac{\tilde{r}(\ell)d(\ell)}{\ell^{\frac{1}{2}+s}} \frac{b(m, \ell)}{m^{1+2s}} \quad (6)$$

**Lemma 2.5.** *We may factor*

$$F(s) = \zeta(2s+1)G(s)H(s), \quad H(s) = \prod_{p \in \mathcal{P}^-} \left(1 + \frac{2\tilde{r}(p)}{p^{\frac{1}{2}+s}}\right).$$

For  $\Re(s) > \frac{-1}{4}$ ,  $|\log G(s)|$  is bounded by an absolute constant.

*Proof.* Straight-forward manipulation yields that for  $\Re(s) > \frac{-1}{4}$ ,  $G(s)$  is given by the absolutely convergent product

$$G(s) = \prod_{p \in \mathcal{P}^-} \left(1 - \frac{1}{p^{2s+2}} \frac{1}{\frac{p+1}{p} + r(p)^2} \frac{1}{1 + \frac{2\tilde{r}(p)}{p^{s+\frac{1}{2}}}}\right) \prod_{\substack{p \notin \mathcal{P}^- \\ \text{odd}}} \left(1 - \frac{1}{p+1} \frac{1}{p^{2s+1}}\right).$$

In this range  $\left| \frac{\tilde{r}(p)}{p^{\frac{1}{2}+s}} \right| = o(1)$  so that the logarithm of the product is bounded independently of  $D$ .  $\square$

We end this section by giving the relatively easy proof that  $\Sigma_2$  is small.

**Proposition 2.6.** *We have  $\Sigma_2 = o(1)$ . In fact,  $\Sigma_2 = S(x)$  and uniformly in  $y \geq 2$  there exist  $C_1, C_2 > 0$  such that*

$$S(y) \ll \log y \exp \left( -C_1 \sqrt{\frac{\log D}{\log \log D}} \right) + \exp \left( -\frac{\log y}{(\log \log D)^2} + C_2 \sqrt{\frac{\log D}{\log \log D}} \right).$$

*Proof.* The conditions on  $\phi$  guarantee that its Mellin transform  $\tilde{\phi}$  has a single simple pole at 0 with residue 1, and that it satisfies the decay property, for each  $A > 0$ , and  $\Im(s) > 1$ ,  $|\tilde{\phi}(s)| \ll |s|^{-A}$ . Therefore, by Mellin inversion,

$$S(y) = \frac{1}{2\pi i} \int_{\frac{1}{\log x} - i\infty}^{\frac{1}{\log x} + i\infty} y^s \tilde{\phi}(s) F(s) ds. \quad (7)$$

Shift the contour to the line  $\Re(s) = \frac{-1}{(\log \log D)^2}$ , writing

$$\begin{aligned} S(y) &= \frac{1}{2\pi i} \int_{|s|=\frac{1}{\log \max(x,y)}} \zeta(2s+1) G(s) H(s) y^s \tilde{\phi}(s) ds \\ &\quad + \frac{1}{2\pi i} \int_{(\frac{-1}{(\log \log D)^2})} \zeta(2s+1) G(s) H(s) y^s \tilde{\phi}(s) ds \end{aligned}$$

where the first term captures the residue at 0. Using the rapid decay of  $\tilde{\phi}$  we find that

$$\begin{aligned} |\Sigma_2| &\ll (\log \max(x, y)) \sup_{|s|=\frac{1}{\log \max(x,y)}} |H(s)| \\ &\quad + \frac{(\log \log D)^2}{\exp(\frac{\log y}{(\log \log D)^2})} \sup_{\Re(s)=\frac{-1}{(\log \log D)^2}} |H(s)|. \end{aligned} \quad (8)$$

Now for  $|s| = \frac{1}{\log \max(x,y)}$  we have, for some  $C > 0$ ,

$$\Re \log \prod_{p \in \mathcal{P}^-} \left( 1 + \frac{2\tilde{r}(p)}{p^{\frac{1}{2}+s}} \right) = (2 + o(1)) \sum_{L^{5\pi/3} \leq p < L^{7\pi/3}} \frac{\cos(\log \frac{p}{L^2}) L}{p \log p} \leq -C \frac{L}{\log L}$$

so that the first term of (8) is  $\leq \log y \exp \left( -C_1 \sqrt{\frac{\log D}{\log \log D}} \right)$ .

When  $\Re(s) = \frac{-1}{(\log \log D)^2}$ ,

$$\Re \log \prod_{p \in \mathcal{P}^-} \left( 1 + \frac{2\tilde{r}(p)}{p^{\frac{1}{2}+s}} \right) \leq (2 + o(1)) \sum_{L^{5\pi/3} \leq p < L^{7\pi/3}} \frac{|\cos(\log \frac{p}{L^2})| L}{p \log p} \ll \frac{L}{\log L}$$

so the second term of (8) is  $\leq \exp \left( C_2 \sqrt{\frac{\log D}{\log \log D}} - \frac{\log y}{(\log \log D)^2} \right)$ .  $\square$



### 3 The large primes

Recall that we let  $\mathcal{P}^+$  be the primes in an interval  $[\frac{B}{4}, B]$  for some  $B$  satisfying  $x^{1-\epsilon} < B \leq x$ . With  $y = \frac{x}{p}$  in mind, recall that we define

$$S(y) = \sum_{(\ell, \mathcal{P}^+)=1} \frac{\tilde{r}(\ell)d(\ell)}{\sqrt{\ell}} \sum_m \frac{b(m, \ell)}{m} \phi\left(\frac{\ell m^2}{y}\right)$$

and set also

$$S^*(y) = \sum_{(\ell, \mathcal{P}^+)=1} \frac{|\tilde{r}(\ell)d(\ell)|}{\sqrt{\ell}} \sum_{m^2 \leq \frac{2y}{\ell}} \frac{b(m, \ell)}{m}.$$

Note the bounds

$$\left| y \frac{d}{dy} S(y) \right| \ll |S^*(y)|, \quad |S^*(y)| \leq \exp\left(O\left(\sqrt{\frac{\log D}{\log \log D}}\right)\right).$$

For the large primes  $p \in [\frac{B}{4}, B]$  we set  $\epsilon_p = -\text{sgn}(S(p))$ , so that

$$\Sigma_1 = \frac{-2}{(\log x)^2} \sum_{B/4 < p \leq B} \frac{|S(x/p)|}{p}.$$

Thus by partial summation against the prime number theorem, using the estimate for  $\frac{d}{dy} S(y)$ , we have

$$\Sigma_1 = \frac{-2}{(\log x)^2} \int_{B/4}^B \frac{|S(x/y)| dy}{y \log y} + o(1).$$

We can make one further reduction. Set  $\psi(x) = \phi(x) - \phi(\frac{x}{2})$  and let

$$\tilde{S}(y) = \sum_{(\ell, \mathcal{P}^+)=1} \frac{\tilde{r}(\ell)d(\ell)}{\sqrt{\ell}} \sum_m \frac{b(m, \ell)}{m} \psi\left(\frac{\ell m^2}{y}\right) = S(y) - S(2y).$$

Then  $|\tilde{S}(y)| \leq |S(y)| + |S(2y)|$  from which it follows

$$\Sigma_1 \leq \frac{-1}{(\log x)^2} \int_{B/2}^B \frac{|\tilde{S}(x/y)| dy}{y \log y} + o(1).$$

Replacing  $\frac{x}{y} =: y$  we complete the proof of Theorem 1.1 by proving the following proposition.

**Proposition 3.1.** *There exists  $2 \leq A \leq U := \exp(\sqrt{\log x}(\log \log x)^2)$ , such that*

$$\int_{A/2}^A |\tilde{S}(y)| \frac{dy}{y} \geq \exp\left((c(a) + o(1)) \sqrt{\frac{\log D}{\log \log D}}\right)$$

For the proof, we appeal to the following modified version of Gallagher's large sieve (see e.g. [1], p. 29).

**Lemma 3.2.** Let  $\psi(x) = \phi(x) - \phi(x/2)$  as above. Define also  $\psi_\sigma(x) = x^\sigma \psi(x)$ . Let  $P(t) = \sum_n a_n n^{-it}$  be a Dirichlet series for which  $\sum |a_n| < \infty$ . There exists a constant  $\alpha = \alpha(\psi) > 0$  such that, uniformly in  $|\sigma| < \frac{1}{2}$ ,

$$\int_{-\alpha}^{\alpha} |P(t)|^2 dt \ll_{\psi} \int_{\frac{1}{2}}^{\infty} \left| \sum_n a_n \psi_{\sigma} \left( \frac{n}{y} \right) \right|^2 \frac{dy}{y}.$$

*Proof.* Note that  $\psi$  has compact support in  $\mathbb{R}^+$  since  $\phi$  has compact support on  $\mathbb{R}$  and  $\phi \equiv 1$  on a neighborhood of zero.

The right hand side is equal to

$$\sum_{n_1, n_2} a_{n_1} \overline{a_{n_2}} \int_0^{\infty} \psi_{\sigma} \left( \frac{n_1}{y} \right) \psi_{\sigma} \left( \frac{n_2}{y} \right) \frac{dy}{y}. \quad (9)$$

Set  $f_{\sigma}(u) = \psi_{\sigma}(e^{-u})$  and set  $H_{\sigma}(x) = \int_{-\infty}^{\infty} f_{\sigma}(u) f_{\sigma}(u+x) du$ . The Fourier transform of  $H_{\sigma}$  is given by  $\hat{H}_{\sigma}(\xi) = |\hat{f}_{\sigma}(\xi)|^2$ , from which it follows that (9) is given by

$$\begin{aligned} \sum_{n_1, n_2} a_{n_1} \overline{a_{n_2}} H_{\sigma} \left( \log \frac{n_1}{n_2} \right) &= \sum_{n_1, n_2} a_{n_1} \overline{a_{n_2}} \int_{-\infty}^{\infty} |\hat{f}_{\sigma}(\xi)|^2 \left( \frac{n_1}{n_2} \right)^{2\pi i \xi} d\xi \\ &\gg \inf_{|\xi| < 2\pi\alpha, |\sigma| \leq \frac{1}{2}} |\hat{f}_{\sigma}(\xi)|^2 \times \int_{-\alpha}^{\alpha} |P(t)|^2 dt. \end{aligned}$$

Let  $f$  have support in  $[-C, C]$ . Then choose  $\alpha \leq \frac{1}{10\pi C}$ , say, to guarantee that  $|\hat{f}_{\sigma}(\xi)|$  is bounded below by a constant depending at most on  $\psi$ .  $\square$

*Proof of Proposition 3.1.* Recall from (6) the definition of the Dirichlet series  $F(s)$ , which we write as  $F(s) = \sum_n \frac{a_n}{n^s}$ . Let  $\sigma = \frac{1}{(\log x)^2}$ . We apply Lemma 3.2 to the Dirichlet series

$$F(\sigma + it) = \sum_n a_{n, \sigma} n^{-it}, \quad a_{n, \sigma} = \frac{a_n}{n^{\sigma}}$$

with function  $\psi_{\sigma}$ . This yields

$$\int_{-\alpha}^{\alpha} |F(\sigma + it)|^2 dt \ll \int_{\frac{1}{2}}^{\infty} |\tilde{S}(y)|^2 \frac{dy}{y^{1+2\sigma}}.$$

Bounding  $|\tilde{S}(y)| \leq |S(y)| + |S(2y)|$ , and applying the bound for  $S(y)$  in Proposition 2.6, we have [recall  $U = \exp(\sqrt{\log x (\log \log x)^2})$ ]

$$\int_{\frac{1}{2}}^{\infty} |\tilde{S}(y)|^2 \frac{dy}{y^{1+2\sigma}} \ll \int_{\frac{1}{2}}^U |\tilde{S}(y)|^2 \frac{dy}{y} + o(1)$$

By dyadic decomposition, we obtain that there is  $A$ ,  $2 \leq A \leq U$  such that

$$\int_{A/2}^A |\tilde{S}(y)| \frac{dy}{y} \gg \frac{1}{\log U \sup_{y < U} |\tilde{S}(y)|} \int_{-\alpha}^{\alpha} |F(\sigma + it)|^2 dt - O(1).$$

Since

$$|\tilde{S}(y)| \ll \log y \sum_{(n, \mathcal{P}^+)=1} \frac{d(n) |\tilde{r}(n)|}{\sqrt{n}},$$

the proof is now completed by the following lemma.  $\square$

**Lemma 3.3.** *For all  $t$  in the interval  $|t - \frac{1}{2\log L}| \leq \frac{1}{(\log \log D)^2}$  we have*

$$\frac{|F(\sigma + it)|^2}{\prod_p \left(1 + \frac{2|\tilde{r}(p)|}{\sqrt{p}}\right)} \geq \exp \left( (c(a) + o(1)) \sqrt{\frac{\log D}{\log \log D}} \right).$$

*Proof.* As in the proof of Proposition 2.6, factor

$$F(\sigma + it) = \zeta(1 + 2\sigma + 2it)G(\sigma + it)H(\sigma + it).$$

Thus

$$\log \frac{|F(\sigma + it)|^2}{\prod_p \left(1 + \frac{2|\tilde{r}(p)|}{\sqrt{p}}\right)} = O(\log \log x) + \sum_{p \in \mathcal{P}^-} \Re \log \left( \prod_{p \in \mathcal{P}^-} \frac{(1 + \frac{2\tilde{r}(p)}{p^{1/2-it}})^2}{1 + \frac{2|\tilde{r}(p)|}{p^{1/2}}} \right)$$

Recall that  $L = \sqrt{\log Y \log \log Y}$ ,  $Y = \min(D^{\delta/2}, x^{1/4})$ , and that

$$\tilde{r}(p) = (1 + o(1)) \frac{L \cos(\frac{\log p}{2\log L})}{\sqrt{p} \log p}.$$

The sum of logarithms is thus

$$2L \sum_{L^{5\pi/3} < p \leq L^{7\pi/3}} \left( \frac{2\Re\{e^{\frac{i\log p}{2\log L}} \cos(\frac{\log p}{2\log L})\} - |\cos(\frac{\log p}{2\log L})|}{p \log p} \right) - o\left(\frac{L}{\log L}\right). \quad (10)$$

Set  $\theta_p = \frac{\log p}{2\log L}$ . Note that  $\theta_p \in [5\pi/6, 7\pi/6]$ . The trigonometric factor above is

$$1 + \cos(\theta_p) + \cos(2\theta_p) = (2 \cos \theta_p + 1) \cos \theta_p \geq \frac{3 - \sqrt{3}}{2}.$$

Thus (10) is

$$\gg L \sum_{L^{5\pi/3} < p \leq L^{7\pi/3}} \frac{1}{p \log p} - o\left(\frac{L}{\log L}\right) \gg \frac{L}{\log L} \gg \sqrt{\frac{\log D}{\log \log D}}.$$

□

## References

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